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ORIGINAL ARTICLE

Multiplicity theorems for scalar periodic problems at resonance with p-Laplacian-like operator

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Abstract In this paper, we study the existence of multiple solutions for nonlinear scalar periodic problems at resonance with p-Laplacian-like operator. Using the Ekeland variational principle a two-solution theorem is obtained and using also a local linking theorem a three-solution theorem is proved.

Keywords Periodic problems · Clarke subdifferential · Resonance · p-Laplacian-like operator · Local linking

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1 Introduction

In this paper, we prove the existence of multiple solutions for the following nonlinear periodic problem:

$$\begin{aligned} &\big(a(t,u'(t))\big)' + \partial j(t,u(t)) \ni 0 \quad \text{for a.a. } t \in (0,T), \\ &u(0) = u(T), \;\; u'(0) = u'(T). \end{aligned} \tag{1.1}$$

Here $(t,y) \longmapsto a(t,y)$ is a set-valued map and $\partial j(t,\zeta)$ is the generalized subdifferential of a generally nonsmooth locally Lipschitz potential $\zeta \longmapsto j(t,\zeta)$. Let $p \in (1,+\infty)$ and consider the Sobolev space

$$W^{1,p}_{\text{per}}((0,T)) \; = \big\{u \in W^{1,p}((0,T)) : \; u(0) = u(T)\big\}.$$

Recall that $W^{1,p}((0,T))$ is embedded into $C([0,T])$ and so the pointwise evaluation at $t=0$ and $t=T$ make sense. For a given $u \in W^{1,p}_{\text{per}}((0,T))$, the multivalued term $\big(a(t,u'(t))\big)'$ is interpreted as follows:

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$$\big(a(t,u'(t))\big)' = \big\{v' \in L^{p'}((0,T)), \quad v(t) \in a(t,u'(t)) \quad \text{for a.a. } t \in (0,T)\big\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Here derivative v' is understood in the sense of distributions. By a solution of problem (1.1) we mean a function $u \in C^1([0,T])$, such that

$$v'(t) \;= u^*(t) \quad \text{for a.a. } t \in (0,T)$$

with $v' \in \big(a(\cdot,u'(\cdot))\big)'$ and $u^* \in L^{p'}((0,T))$, $u^*(t) \in \partial j(t,u(t))$ for almost all $t \in (0,T)$.

Our hypotheses on the set-valued map $a(t,y)$, include as a special case the scalar p-Laplacian differential operator. Recently there has been increasing interest for second-order scalar periodic differential equations involving the p-Laplacian differential operator. We mention the works of Dang and Oppenheimer [6], Denkowski et al. [8], Del Pino et al. [7], Fabry and Fayyad [9], Gasiński and Papageorgiou [10,11], Guo [13] and Papageorgiou and Papageorgiou [19]. Most of the aforementioned works prove existence theorems. Multiplicity results were proved only by Del Pino et al., Denkowski et al., Gasinski-Papageorgiou and Papageorgiou–Papageorgiou. In all these works the differential operator is the scalar p-Laplacian and the first and third assume a smooth potential (i.e. $j(t,\cdot) \in C^1(\mathbb{R})$), while in Gasiński–Papageorgiou the potential $j(t,\cdot)$ is in general nonsmooth. In Del Pino et al. the method of the proof uses degree theory and the time map. In Gasiński–Papageorgiou and Papageorgiou–Papageorgiou, the approach is variational using local linking (Gasiński–Papageorgiou) or the so-called second deformation theorem (Papageorgiou–Papageorgiou). All these works prove the existence of two solutions. For other periodic multiple solutions of hemivariational inequalities, we refer to Adly and Motreanu [1] and Motreanu and Rădulescu [18].

Our approach in the paper is variational and uses the critical point theory for locally Lipschitz functions (see Chang [4] and Kourogenis and Papageorgiou [14]). We also prove a "three solution theorem". This is done for a so-called "strongly resonant" problem (terminology coined by Bartolo et al. [2]). None of the previous works mentioned above examined such problems. The main difficulty that such problems exhibit is a partial lack of compactness (see Lemma 3.4 below).

In the next section, we recall basic definitions and notions needed in what follows. Section 3 contains the theorem on the existence of two solutions of problem (1.1). In Sect. 4 we proof the theorem on the existence of there solutions of problem (1.1).

2 Mathematical background

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|$ we denote the norm in X, by $\|\cdot\|_*$ the norm in X^*, and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X,X^*). A function $\varphi : X \longmapsto \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$, there exists a neighbourhood U of x and a constant $k > 0$ (depending on U), such that $|\varphi(z) - \varphi(y)| \le k\|z-y\|$ for all $z,y \in U$. It is well known that a convex, lower semicontinuous and proper (i.e. not identically $+\infty$) function g: $X \longmapsto \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\text{dom} g = \{x \in X : \; g(x) < +\infty\}$. For a locally Lipschitz function φ: $X \longmapsto \mathbb{R}$, we define the generalized directional derivative at $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\varphi(x'+th) - \varphi(x')}{t}.$$

The function $X \ni h \longmapsto \varphi^0(x;h) \in \mathbb{R}$ is sublinear, continuous and so from the Hahn–Banach theorem it follows that $\varphi^0(x;\cdot)$ is the support function of a nonempty, convex and w^*-compact set, defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \text{ for all } h \in X\}.$$

The set $\partial\varphi(x)$ is called *generalized* or *Clarke* subdifferential of φ at x. If $\varphi\colon X \longmapsto \mathbb{R}$ is also convex, then the subdifferential of φ in the sense of convex analysis coincides with the generalized subdifferential introduced above. If φ is strictly differentiable at x (in particular if φ is continuously Gâteaux differentiable at x), then $\partial\varphi(x) = \{\varphi'(x)\}$. If $\varphi, \psi\colon X \longmapsto \mathbb{R}$ are locally Lipschitz functions, then $\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$ and $\partial(t\varphi)(x) = t\partial\varphi(x)$ for all $t \in \mathbb{R}$ and all $x \in X$.

Let $\varphi\colon X \longmapsto \mathbb{R}$ be a locally Lipschitz function on a Banach space X. A point $x \in X$ is said to be a *critical point* of φ, if $0 \in \partial\varphi(x)$. If $x \in X$ is a critical point of φ, then the value $c = \varphi(x)$ is called a *critical value* of φ. It is easy to see that, if $x \in X$ is a local extremum of φ, then $0 \in \partial\varphi(x)$. Moreover, the multifunction $X \ni x \longmapsto \partial\varphi(x) \in 2^{X^*}$ is *upper semicontinuous*, where the space X^* is equipped with the w^*-topology, i.e. for any w^*-open set $U \subseteq X^*$, the set $\{x \in X : \quad \partial\varphi(x) \subseteq U\}$ is open in X. For more details on the generalized subdifferential we refer to the book of Clarke [5, Chap. 2].

In the classical (smooth) critical point theory, crucial role plays a compactness type condition, known as the *Palais–Smale condition*. When the function is only locally Lipschitz, this condition takes the following form (introduced by Chang [4, Definition 2, p 113])

A locally Lipschitz function $\varphi\colon X \longmapsto \mathbb{R}$ satisfies the *nonsmooth Palais-Smale condition*, if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\sup\{\varphi(x_n) : \quad n \geq 1\} < +\infty$$

and

$$m^{\varphi}(x_n) = \inf\{\|x^*\|_* : x^* \in \partial\varphi(x_n)\} \longrightarrow 0 \quad \text{as} \quad n \to +\infty,$$

has a strongly convergent subsequence.

If $\varphi \in \mathcal{C}^1(X)$, then $\partial\varphi(x_n) = \{\varphi'(x_n)\}$ and so we see that the above definition of the Palais–Smale condition coincides with the classical one.

We will also use a weaker form of the Palais–Smale condition, which for the smooth functions was first introduced by Cerami [3]. In our nonsmooth setting this condition takes the following form

A locally Lipschitz function $\varphi\colon X \longmapsto \mathbb{R}$ satisfies the *nonsmooth Cerami condition*, if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\sup\{\varphi(x_n) : n \geq 1\} < +\infty$$

and

$$(1 + \|x_n\|)m^{\varphi}(x_n) \longrightarrow 0 \quad \text{as } n \to +\infty,$$

has a strongly convergent subsequence.

In our hypotheses, we will use the first nonzero eigenvalue λ_1 of the negative p-Laplacian $-\Delta_p u = -\left(|u'|^{p-2}u'\right)'$ with periodic boundary condition. So we consider the following quasilinear eigenvalue problem:

$$\begin{aligned} &-\left(|u'(t)|^{p-2}u'(t)\right)' = \lambda|u(t)|^{p-2}u(t) \quad \text{for a.a. } t \in (0,T) \\ &u(0) = u(T), \;\; u'(0) = uv(T). \end{aligned} \tag{2.1}$$

It is well-known that $\lambda_0 = 0$ is an eigenvalue of (2.1) and is simple and isolated. So, if $\lambda_1 = \inf\{\lambda > 0 : \lambda \text{is an eigenvalue of} - \Delta_p\}$, then $\lambda_1 > 0$ and

$$\|u'\|_p^p \geq \lambda_1 \|u\|_p^p, \quad \forall u \in V, \tag{2.2}$$

where $V = \left\{u \in W^{1,p}_{\text{per}}((0,T)) : \int_0^T |u(t)|^{p-2}u(t)\mathrm{d}t = 0\right\}$ (see Mawhin [17, Corollary 9.3, p 60]).

We will use the generalized Ekeland variational principle (see e.g. Gasiński and Papageorgiou [12, Corollary 1.4.7, p 91]), in the following form

Theorem 2.1 *If (X, d_X) is a complete metric space, $\varphi\colon X \longrightarrow \overline{\mathbb{R}}$ is proper, lower semicontinuous and bounded below, $\varepsilon, \lambda > 0$ and $x_0 \in X$ is such that*

$$\varphi(x_0) \leq \inf_X \varphi + \varepsilon,$$

then there exists $x_\lambda \in X$, such that

$$\begin{aligned} &\varphi(x_\lambda) \leq \varphi(x_0), \qquad \mathrm{d}(x_\lambda, x_0) \leq \lambda, \\ &\varphi(x_\lambda) \leq \; \varphi(x) + \frac{\varepsilon}{\lambda}\mathrm{d}(x, x_\lambda), \quad \forall x \in X. \end{aligned}$$

The next result is due to Szulkin [20, Lemma 3.1, p 81].

Theorem 2.2 *If X is a Banach space, $\chi\colon X \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous, convex function with $\chi(0) = 0$ and*

$$-\|h\|_X \;\leq \chi(h), \quad \forall h \in X,$$

then there exists $u^* \in X^*$, such that $\|u^*\|_{X^*} \leq 1$ and

$$\langle u^*, h\rangle \;\leq\; \chi(h), \quad \forall h \in X.$$

In the three-solution result we will use the notion of linking, which plays a crucial role in critical point theory (classical and nonsmooth alike). Suppose that X is a Hausdorff topological space and E_1 and D are nonempty subsets of X. We say that the sets E_1 and D *link (homotopically)* in X if $E_1 \cap D = \emptyset$ and there exists a set $E \subseteq X$, such that $E_1 \subseteq E$ and for any continuous function $\vartheta\colon E \longrightarrow X$, such that $\vartheta|_{E_1} = id_{E_1}$, we have $\vartheta(E) \cap D \neq \emptyset$.

Using this notion, Kourogenis and Papageorgiou [14] proved the following abstract minimax principle (see also Gasiński and Papageorgiou [12, Theorem 2.1.2, p139] for a more general version).

Theorem 2.3 *If X is a reflexive Banach space, E_1 and D are nonempty subsets of X with D closed, E_1 and D link in X, $\varphi\colon X \to \mathbb{R}$ is locally Lipschitz, satisfies the nonsmooth Cerami condition, $\sup\limits_{E_1} \varphi < \inf\limits_D \varphi$ and*

$$c \;=\; \inf_{\eta \in \Gamma} \sup_{v \in E} \varphi(\eta(v)),$$

where

$$\Gamma = \{\eta \in C(E;X) : \eta|_{E_1} = id_{E_1}\}$$

and $E \supseteq E_1$ is as in the definition of linking sets,
then $c \geq \inf_D \varphi$ and c is a critical value of φ, i.e. there exists a critical point $x_0 \in X$ of φ such that $\varphi(x_0) = c$. Moreover, if $c = \inf_D \varphi$, then $x_0 \in D$.

3 Existence of two solutions

The precise hypotheses on the data of (1.1) are the following:

$H(a)$ $a(t,y) = \partial G(t,y)$, where G: $(0,T) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a functional, such that

(1) the function $(t,y) \longrightarrow G(t,y)$ is continuous;
(2) for every $t \in (0,T)$, the function $y \longmapsto G(t,y)$ is strictly convex, $G(t,0) = 0$ for all $t \in (0,T)$ and $\partial G(0,\cdot) = \partial G(T,\cdot)$;
(3) for all $t \in (0,T)$, all $y \in \mathbb{R}$ and all $v^* \in a(t,y) = \partial G(t,y)$, we have

$$|v^*| \leq a_1(t) + c_1|y|^{p-1},$$

with $a_1 \in L^{p'}((0,T))_+$ (where $\frac{1}{p} + \frac{1}{p'} = 1$), $c_1 > 0$;
(4) for all $t \in (0,T)$, all $y \in \mathbb{R}$ and all $v^* \in a(t,y)$, we have

$$v^* y \leq pG(t,y);$$

(5) for all $t \in (0,T)$ and all $y \in \mathbb{R}$, we have

$$c_0|y|^p \leq G(t,y),$$

for some $c_0 > 0$.

Remark 3.1 Suppose that $\beta \in C_{\text{per}}([0,T])$, $\beta \geq \gamma > 0$ for all $t \in (0,T)$ and $G(t,y) = \frac{1}{p}\beta(t)|y|^p$. Then

$$a(t,y) = \partial G(t,y) = \beta(t)|y|^{p-2}y$$

satisfies hypotheses $H(a)$ and the resulting differential operator is a weighted p-Laplacian. If $\beta \equiv 1$, then we have the p-Laplacian. We remark that hypotheses $H(a)$ do not require that the differential operator is homogeneous. Such single valued operators independent of $t \in (0,T)$ were considered by Manásevich and Mawhin [15] and Mawhin [16]. However, in these works the problem is vectorial and no growth restriction is imposed on the map $y \longmapsto a(y)$.

Another possibility of G is the following

$$G(t,y) = \frac{\beta(t)}{p}\left[(1+y^2)^{\frac{p}{2}} - 1\right]$$

with $p > 1$ and $\beta \in C_{\text{per}}([0,T])$, $\beta(t) \geq \gamma > 0$ for all $t \in (0,T)$.

One more possibility of G is the following

$$G(t,y) = \frac{\beta(t)}{p}\left[(1+|y|)^p - 1\right],$$

where $p > 1$ and β are as above. In this case, the map a is really multivalued and it still satisfies hypotheses $H(a)$.

$H(j)_1$ j: $(0,T) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(1) for every $\zeta \in \mathbb{R}$, the function $t \longrightarrow j(t,\zeta)$ is measurable;
(2) for almost all $t \in (0,T)$, the function $\zeta \longmapsto j(t,\zeta)$ is locally Lipschitz with $L^{p'}((0,T))_+$-Lipschitz constant;
(3) for every $M > 0$, there exists $\widehat{a}_M \in L^1((0,T))_+$, such that for almost all $t \in (0,T)$, all $|\zeta| \leq M$ and all $u^* \in \partial j(t,\zeta)$, we have $|u^*| \leq \widehat{a}_M(t)$;
(4) there exist $j_\pm \in L^1((0,T))$, such that

$$\lim_{\zeta \to \pm\infty} j(t,\zeta) = j_\pm(t),$$

uniformly for almost all $t \in (0,T)$ and $\int_0^T j_\pm(t)\mathrm{d}t \leq 0$;
(5) there exists $\delta > 0$, such that for almost all $t \in (0,T)$ and all $|\zeta| \leq \delta$, we have $j(t,\zeta) \geq 0$ (local sign condition);
(6) for almost all $t \in (0,T)$ and all $\zeta \in \mathbb{R}$, we have

$$j(t,\zeta) \ \leq c_0\lambda_1|\zeta|^p$$

with $c_0 > 0$ as in hypothesis $H(a)(5)$ and $\lambda_1 > 0$ being the first nonzero eigenvalue of the negative p-Laplacian with periodic boundary condition.

Remark 3.2 Hypothesis $H(j)_1(4)$ classifies the problem as strongly resonant. Hypotheses $H(j)_1(5)$ and (6) imply that $j(t,0) = 0$ for almost all $t \in (0,T)$.

We consider the nonlinear operator A: $W^{1,p}_{\mathrm{per}}((0,T)) \longrightarrow 2^{W^{1,p}_{\mathrm{per}}((0,T))^*}$, defined by

$$A(u) = \Big\{ v^* \in W^{1,p}_{\mathrm{per}}((0,T))^* : \text{there exists } v \in S^{p'}_{a(\cdot,u'(\cdot))} \text{ such that}$$
$$\text{for all} y \in W^{1,p}_{\mathrm{per}}((0,T)) : \ \langle v^*, y\rangle = \int_0^T v(t)y'(t)\mathrm{d}t \Big\}.$$

Hence

$$A(u) = \big\{ -v' : \ v \in S^{p'}_{a(\cdot,u'(\cdot))} \big\}$$

(the derivative taken in the sense of the distributions). Clearly for every $u \in W^{1,p}_{\mathrm{per}}((0,T))$, the set $A(u) \subseteq W^{1,p}_{\mathrm{per}}((0,T))^*$ is nonempty, convex and w-compact. Moreover, the function $u \longmapsto A(u)$ is monotone, thus in fact maximal monotone (see Gasiński and Papageorgiou [12, Proposition 1.4.6, p 74]).

Lemma 3.3 *Let hypotheses $H(a)$ hold. If $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}((0,T))$ is a sequence, such that*

$$u_n \longrightarrow u \quad \textit{weakly in } W^{1,p}_{\mathrm{per}}((0,T)), \tag{3.1}$$

$$v_n^* \in A(u_n) \quad \forall n \geq 1$$

and

$$\limsup_{n\to+\infty}\langle v_n^*, u_n - u\rangle \le 0, \tag{3.2}$$

then

$$u_n \longrightarrow u \quad in\ W^{1,p}_{\text{per}}((0,T)). \tag{3.3}$$

Proof Let $\{u_n\}_{n\geq1} \subseteq W^{1,p}_{\text{per}}((0,T))$ and $\{v_n^*\}_{n\geq1} \subseteq W^{1,p}_{\text{per}}((0,T))^*$ be sequences as postulated in the assumptions of the lemma. Then, we have

$$\langle v_n^*, u_n - u\rangle = \int_0^T v_n(t)(u_n - u)'(t)\mathrm{dt} \quad \forall n \geq 1,$$

with $_n \in S^{p'}_{a(\cdot,u_n'(\cdot))}$. By virtue of hypothesis $H(a)(3)$, we see that the sequence $\{v_n\}_{n\geq1} \subseteq L^{p'}((0,T))$ is bounded. So by passing to a subsequence if necessary, we may assume that

$$v_n \longrightarrow v \quad \text{weakly in } L^{p'}((0,T)) \tag{3.4}$$

for some $v \in L^{p'}((0,T))$.

We claim that $v \in S^{p'}_{a(\cdot,u'(\cdot))}$. To this end let $y \in W^{1,p}_{\text{per}}((0,T))$ and $w^* \in A(y)$. From the definition of the operator A, we know that we can find $w \in S^{p'}_{a(\cdot,y'(\cdot))}$, such that

$$\langle w^*, z\rangle = \int_0^T w(t)z'(t)\mathrm{d}t, \quad \forall z \in W^{1,p}_{\text{per}}((0,T)).$$

Since $a(t,\zeta) = \partial G(t,\zeta)$, the operator $\zeta \longrightarrow a(t,\zeta)$ is maximal monotone and so, we have

$$\begin{aligned}
0 &\leq \int_0^T (v_n(t) - w(t))(u_n'(t) - y'(t))\mathrm{d}t \\
&= \int_0^T v_n(t)(u_n' - u')(t)\mathrm{d}t + \int_0^T v_n(t)(u' - y')(t)\mathrm{d}t \\
&\quad - \int_0^T w(t)(u_n' - y')(t)\mathrm{d}t \\
&= \langle v_n^*, u_n - u\rangle + \int_0^T v_n(t)(u' - y')(t)\mathrm{d}t - \int_0^T w(t)(u_n' - y')(t)\mathrm{d}t.
\end{aligned} \tag{3.5}$$

From (3.1), (3.2) and (3.4), if we pass to the limit as $n \to +\infty$ in (3.5), we obtain

$$0 = \int_0^T (v(t) - w(t))(u'(t) - y'(t))\mathrm{d}t = \langle v^* - w^*, u - y\rangle$$

with $v^* = -v'$. But the pair $(y, w^*) \in \mathrm{Gr}\,A$ was arbitrary and we know that the operator A is maximal monotone. So it follows that $(u, v^*) \in \mathrm{Gr}\,A$, i.e. $v^* \in A(u)$ and

$$\langle v^*, z\rangle = \int_0^T \overline{v}(t)z'(t)\mathrm{d}t, \quad \forall z \in W^{1,p}_{\text{per}}((0,T))$$

for some $\overline{v} \in S^{p'}_{a(\cdot,u'(\cdot))}$ and so $v = \overline{v} \in S^{p'}_{a(\cdot,u'(\cdot))}$.

Because of the hypotheses, we have

$$\limsup_{n\to+\infty}\langle v_n^* - v^*, u_n - u\rangle \leq 0. \tag{3.6}$$

On the other hand by virtue of the monotonicity of $a(t,\cdot)$, we have

$$\liminf_{n\to+\infty}\langle v_n^*-v^*,u_n-u\rangle = \liminf_{n\to+\infty}\int_0^T (v_n(t)-v(t))(u_n'(t)-u'(t))\mathrm{d}t \geq 0. \qquad (3.7)$$

Comparing (3.6) and (3.7), we infer that

$$\langle v_n^*-v^*,u_n-u\rangle = \int_0^T (v_n(t)-v(t))(u_n'(t)-u'(t))\mathrm{d}t \longrightarrow 0.$$

Because of the monotonicity of $a(t,\cdot)=\partial G(t,\cdot)$, the integrand

$$\beta_n(t) = (v_n-v)(t)(u_n'-u')(t) \longrightarrow 0 \quad \text{for a.a. } t\in(0,T). \qquad (3.8)$$

So we have

$$\beta_n(t) \longrightarrow 0 \quad \text{for a.a. } t\in(0,T)$$

and

$$|\beta_n(t)| \leq k_1(t) \quad \text{for a.a. } t\in(0,T) \text{ and all } n\geq 1,$$

with $k_1\in L^1((0,T))_+$. For all $(t,y)\in(0,T)\times\mathbb{R}$ and all $v\in a(t,y)$, from the definition of the convex subdifferential, we have

$$v(-y) \leq G(t,0)-G(t,y) = -G(t,y)$$

so, from hypothesis $H(a)(5)$, we get

$$vy \geq G(t,y) \geq c_0|y|^p. \qquad (3.9)$$

Using hypothesis $H(a)(3)$ and (3.9), for all $t\in(0,T)\setminus N$, with $|N|_1=0$, we have

$$\begin{aligned} k_1(t)\geq\beta_n(t) &= (v_n-v)(t)(u_n'-u')(t)\\ &\geq c_0\big[|u_n'(t)|^p+|u'(t)|^p\big]-|u_n'(t)|^p\big(a_1(t)+c_1|u'(t)|^{p-1}\big)\\ &\quad -|u'(t)|\big(a_1(t)+c_1|u_n'(t)|^{p-1}\big). \end{aligned} \qquad (3.10)$$

From (3.10), it follows that for all $t\in(0,T)\setminus N$, the sequence $\{u_n'(t)\}_{n\geq1}\subseteq\mathbb{R}$ is bounded. So by passing to a subsequence (depending in general on $t\in(0,T)\setminus N$), we may assume that

$$u_n'(t) \longrightarrow \widehat{u}(t) \quad \text{in } \mathbb{R}.$$

We fix $t\in(0,T)\setminus N$ and select $f_n(t)\in a(t,\widehat{u}(t))$, such that

$$\big|v_n(t)-f_n(t)\big| = \mathrm{d}\big(v_n(t),a(t,\widehat{u}(t))\big) \leq h^*\big(a(t,u_n'(t)),a(t,\widehat{u}(t))\big),$$

with h^* being the Hausdorff distance of sets (see Gasiński and Papageorgiou [12, Definition 1.2.4, p 18]). Note that $\{f_n(t)\}_{n\geq1}\subseteq a(t,\widehat{u}(t))\in P_{kc}(\mathbb{R})$ and so by passing to a subsequence if necessary, we may assume that

$$f_n(t) \longrightarrow f(t)\in a(t,\widehat{u}(t)).$$

Because $a(t,\cdot)$ is maximal monotone, it is upper semicontinuous (see Gasiński and Papageorgiou [12, Proposition 1.4.5, p 73]) and also h-upper semicontinuous (see Gasiński and Papageorgiou [12, Proposition 1.2.8, p 19]). Therefore, we have

$$h^*\big(a(t,u_n'(t)),a(t,\widehat{u}(t))\big) \longrightarrow 0$$

so

$$v_n(t) \longrightarrow f(t) \quad \text{for all } t \in (0,T)\backslash N.$$

Because of (3.8), in the limit as $n \to +\infty$, we have

$$(f(t) - v(t))(\widehat{u}(t) - u'(t)) = 0, \quad \forall t \in (0,T)\backslash N. \tag{3.11}$$

By hypothesis, we have that $a(t,\cdot) \in \partial G(t,\cdot)$ and the function $y \longmapsto G(t,y)$ is strictly convex. Therefore the operator $y \longmapsto a(t,y)$ is strictly monotone. Since $f(t) \in a(t,\widehat{u}(t))$ and $v(t) \in a(t,u'(t))$ for all $t \in (0,T)\backslash N$, from (3.11), we infer that $\widehat{u}(t) = u'(t)$ for all $t \in (0,T)\backslash N$. Therefore, we have

$$u_n'(t) \longrightarrow u'(t) \quad \text{for a.a. } t \in (0,T) \tag{3.12}$$

and from (3.1), also

$$u_n' \longrightarrow u' \quad \text{weakly in } L^p((0,T)). \tag{3.13}$$

From (3.10), we see that

$$\begin{aligned} c_0|u_n'(t)|^p \le k_1(t) + c_0|u'(t)|^p + |u'(t)|\big(a_1(t) + c_1|u_n'(t)|^{p-1}\big) \\ + |u_n'(t)|\big(a_1(t) + c_1|u'(t)|^{p-1}\big). \end{aligned}$$

Using Young's inequality with $\varepsilon > 0$, we obtain

$$\begin{aligned} c_0|u_n'(t)|^p \le k_1(t) + c_0|u'(t)|^p + a_1(t)|u'(t)| + \frac{c_1^p}{\varepsilon p}|u'(t)|^p + \frac{\varepsilon}{p'}|u_n'(t)|^p \\ + a_1(t)|u_n'(t)| + \frac{\varepsilon}{p}|u_n'(t)|^p + \frac{c_1^{p'}}{\varepsilon p'}|u'(t)|^p. \end{aligned} \tag{3.14}$$

If we choose $\varepsilon < c_0$ (recall that $\frac{1}{p} + \frac{1}{p'} = 1$), from (3.14), it follows that the sequence $\{|u_n'(\cdot)|^p\} \subseteq L^1((0,T))$ is uniformly integrable. Because of (3.12), (3.13) and Vitali's Theorem (see e.g. Gasiński and Papageorgiou [12, Theorem A.2.1, p 715]), we have that

$$\|u_n'\|_p \longrightarrow \|u'\|_p. \tag{3.15}$$

Combining (3.12), (3.13) and (3.15) and using the Kadec–Klee property (see Gasiński and Papageorgiou [12, Remark A.3.11, p722]), we have that

$$u_n' \longrightarrow u' \quad \text{in } L^p((0,T))$$

so finally (3.3) holds. □

We consider the energy functional $\varphi\colon W^{1,p}_{\text{per}}((0,T)) \longrightarrow \mathbb{R}$, defined by

$$\varphi(u) = \int_0^T G(t,u'(t))\mathrm{d}t - \int_0^T j(t,u(t))\mathrm{d}t.$$

We know that φ is locally Lipschitz (see Gasiński and Papageorgiou [12, Theorem 1.3.10, p 59]).

The next lemma illustrates the partial lack of compactness which characterizes strongly resonant problems.

Lemma 3.4 *If hypotheses H(a) and $H(j)_1$ hold, then φ satisfies the nonsmooth Cerami condition at any level $c \neq -\int_0^T j_\pm(t)\mathrm{d}t$.*

Proof Let $c \neq -\int_0^T j_\pm(t)\mathrm{d}t$ and let $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}((0,T))$ be a sequence, such that

$$\varphi(u_n) \longrightarrow c \quad \text{and} \quad (1+\|u_n\|)m^{\varphi}(u_n) \longrightarrow 0. \tag{3.16}$$

Let $w_n^* \in \partial\varphi(u_n)$ be such that $m^{\varphi}(u_n) = \|w_n^*\|_*$ for all $n \geq 1$. The existence of such an element follows from the weak lower semicontinuity of the norm functional in $W^{1,p}_{\mathrm{per}}((0,T))^*$ and from the weak compactness of $\partial\varphi(u_n) \subseteq W^{1,p}_{\mathrm{per}}((0,T))^*$.

Let I_G: $L^p((0,T)) \longrightarrow \mathbb{R}$ be the integral functional, defined by

$$I_G(y) = \int_0^T G(t,y(t))\mathrm{d}t.$$

We know that I_G is continuous, convex. Let $D \in \mathcal{L}\big(W^{1,p}_{\mathrm{per}}((0,T)); L^p((0,T))\big)$ be defined by

$$Du = \frac{d}{\mathrm{d}t}u.$$

We have

$$\int_0^T G(t,u'(t))\mathrm{d}t = (I_G \circ D)(u), \quad \forall u \in W^{1,p}_{\mathrm{per}}((0,T))$$

so

$$\partial(I_G \circ D)(u) = -D^*\partial I_G(u') = -\frac{d}{\mathrm{d}t}\partial I_G(u'), \quad \forall u \in W^{1,p}_{\mathrm{per}}((0,T))$$

and finally

$$\partial(I_G \circ D)(u) = A(u)$$

(see Gasiński and Papageorgiou [12, Proposition 1.3.15, p 54 and Remark 1.3.6, p 55]). Then

$$w_n^* = v_n^* - u_n^*, \quad \forall n \geq 1, \tag{3.17}$$

with $v_n^* \in A(u_n)$ and $u_n^* \in S^{p'}_{\partial j(\cdot,u_n(\cdot))}$. We claim that the sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}((0,T))$ is bounded. Suppose that this is not true. By passing to a subsequence if necessary, we may assume that

$$\|u_n\| \longrightarrow +\infty.$$

Let $y_n = \frac{u_n}{\|u_n\|}$ for all $n \geq 1$. We may assume that

$$\begin{aligned} y_n &\longrightarrow y \quad \text{weakly in } W^{1,p}_{\mathrm{per}}((0,T)),\\ y_n &\longrightarrow y \quad \text{in } C([0,T]) \end{aligned}$$

(recall that the embedding $W^{1,p}_{\mathrm{per}}((0,T)) \subseteq C([0,T])$ is compact). From the choice of the sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}((0,T))$, we have

$$\frac{\varphi(u_n)}{\|u_n\|^p} = \int_0^T \frac{G(t,u_n'(t))}{\|u_n\|^p}\mathrm{d}t - \int_0^T \frac{j(t,u_n(t))}{\|u_n\|^p}\mathrm{d}t \leq \frac{M_1}{\|u_n\|^p} \tag{3.18}$$

for some $M_1 > 0$. Because of hypothesis $H(a)(5)$, for almost all $t \in (0, T)$ and all $n \geq 1$, we have

$$c_0|u_n'(t)|^p \ \leq \ G(t, u_n'(t))$$

so

$$c_0|y_n'(t)|^p \ \leq \ \frac{G(t, u_n'(t))}{\|u_n\|^p}. \tag{3.19}$$

Moreover, by virtue of hypothesis $H(j)_1(4)$, we can find $M_2 > 0$, such that for almost all $t \in (0, T)$ and all $|\zeta| > M_2$, we have

$$|j(t, \zeta)| \ \leq \ \max\left\{|j_+(t)|, |j_-(t)|\right\} + 1. \tag{3.20}$$

On the other hand from hypothesis $H(j)_1(3)$ and the mean value theorem for locally Lipschitz functions (see e.g. Gasiński and Papageorgiou [12, Proposition 1.3.14, p53]), we see that for almost all $|\zeta| \leq M_2$, we have

$$|j(t, \zeta)| \ \leq \ \beta_1(t) \tag{3.21}$$

with $\beta_1 \in L^1((0, T))_+$. From (3.20) and (3.21), we conclude that for almost all $t \in (0, T)$ and all $\zeta \in \mathbb{R}$, we have

$$|j(t, \zeta)| \ \leq \ \beta_2(t)$$

with $\beta_2 \in L^1((0, T))_+$. So we have

$$\left|\int_0^T \frac{j(t, u_n(t))}{\|u_n\|^p} \mathrm{d}t\right| \ \leq \ \frac{\|\beta_2\|_1}{\|u_n\|^p} \ \longrightarrow \ 0. \tag{3.22}$$

Passing to the limit as $n \to +\infty$ in (3.18) and using (3.19) and (3.22), we obtain

$$c_0\|y'\|_p^p \ \leq \ c_0 \liminf_{n\to+\infty} \|y_n'\|_p^p \ \leq \ 0$$

so $y \equiv \xi \in \mathbb{R}$.

If $\xi = 0$, then from (3.18) and (3.19), we have

$$c_0\|y_n'\|_p^p \ \leq \ \frac{M_1}{\|u_n\|^p} + \int_0^T \frac{j(t, u_n(t))}{\|u_n\|^p} \mathrm{d}t,$$

so

$$y_n' \ \longrightarrow \ 0 \quad \text{in } L^p((0, T))$$

and thus

$$y_n \ \longrightarrow \ 0 \quad \text{in } W^{1,p}_{\text{per}}((0, T)),$$

a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. So $\xi \neq 0$. Suppose that $\xi > 0$. Then

$$u_n(t) \ \longrightarrow \ +\infty, \quad \forall t \in (0, T).$$

In fact we claim that this convergence is uniform in $t \in (0, T)$. To this end let $\delta' \in (0, \xi)$. Since $y_n \longrightarrow \xi$ in $C([0, T])$, we can find $n_0 = n_0(\delta') \geq 1$, such that for all $n \geq n_0$ and all $t \in (0, T)$, we have

$$|y_n(t) - \xi| \ < \ \delta'$$

so

$$0 < \delta_1 = \xi - \delta' \leq y_n(t)$$

(hence $u_n(t) > 0$ for all $n \geq n_0$ and all $t \in (0, T)$).

Moreover, since $\|u_n\| \longrightarrow +\infty$, for a given $\eta > 0$, we can find $n_1 = n_1(\eta) \geq n_0$, such that

$$\|u_n\| \geq \eta > 0, \quad \forall n \geq n_1.$$

For all $n \geq n_1$ and all $t \in (0, T)$, we have

$$\frac{u_n(t)}{\eta} \geq \frac{u_n(t)}{\|u_n\|} = y_n(t) \geq \delta_1 > 0$$

so

$$u_n(t) \geq \eta\delta_1 > 0, \quad \forall t \in (0, T), \ n \geq n_1.$$

Because $\eta > 0$ was arbitrary, we conclude that

$$\min_{t\in[0,T]} u_n(t) \longrightarrow +\infty.$$

Using this fact in conjunction with hypothesis $H(j)_1(4)$, we see that for a given $\varepsilon > 0$, we can find $n_2 = n_2(\varepsilon) \geq 1$, such that for almost all $t \in (0, T)$ and all $n \geq n_2$, we have

$$j_+(t) - \varepsilon \leq j(t, u_n(t)) \leq j_+(t) + \varepsilon$$

so

$$\int_0^t j(t, u_n(t))\mathrm{d}t \longrightarrow \int_0^T j_+(t)\mathrm{d}t. \tag{3.23}$$

Recall that $\varphi(u_n) \longrightarrow c$. For a given $\varepsilon > 0$, we can find $n_3 = n_3(\varepsilon) \geq n_2$, such that

$$|\varphi(u_n) - c| \leq \varepsilon, \quad \forall n \geq n_3$$

so

$$c - \varepsilon \leq \varphi(u_n) = \int_0^T G(t, u_n'(t))\mathrm{d}t - \int_0^T j(t, u_n(t))\mathrm{d}t \leq c + \varepsilon. \tag{3.24}$$

From the choice of the sequence $\{u_n\} \subseteq W^{1,p}_{\mathrm{per}}((0, T))$ (see (3.16) and (3.17)), we have

$$\left|\langle v_n^*, u_n\rangle - \int_0^T u_n^*(t)u_n(t)\mathrm{d}t\right| \leq \varepsilon_n$$

with $\varepsilon_n \searrow 0$, so

$$\left|\int_0^T v_n(t)u_n'(t)\mathrm{d}t - \int_0^T u_n^*(t)u_n(t)\mathrm{d}t\right| \leq \varepsilon_n \tag{3.25}$$

with $v_n \in S^{p'}_{a(\cdot,u_n'(\cdot))}$. From the definition of the generalized subdifferential, for almost all $t \in (0, T)$ and all $n \geq 1$, we have

$$\begin{aligned} u_n^*(t)u_n(t) &\leq j^0(t, u_n(t); u_n(t)) \\ &= \limsup_{\substack{z_m^n \to u_n(t) \\ \varepsilon \searrow 0}} \frac{j(t, z_m^n + \varepsilon u_n(t)) - j(t, z_m^n)}{\varepsilon}. \end{aligned} \tag{3.26}$$

Because $u_n(t) \longrightarrow +\infty$ (uniformly in $t \in (0, T)$), up to a subsequence, we must have $z^n_{m(n)} \longrightarrow +\infty$ as $n \to +\infty$ and so by virtue of hypothesis $H(j)_1(4)$, for a given $\varepsilon > 0$, we can find $n_4 = n_4(\varepsilon) \geq 1$, such that

$$j_+(t) - \frac{\varepsilon^2}{2} \leq j(t, z^n_{m(n)} + \varepsilon u_n(t)) \leq j_+(t) + \frac{\varepsilon^2}{2}, \quad \forall n \geq n_4 \tag{3.27}$$

and

$$j_+(t) - \frac{\varepsilon^2}{2} \leq j(t, z^n_{m(n)}) \leq j_+(t) + \frac{\varepsilon^2}{2}, \quad \forall n \geq n_4. \tag{3.28}$$

Using (3.27) and (3.28) in (3.26), we see that for almost all $t \in (0, T)$ and all $n \geq n_4$, we have

$$|u^*_n(t)u_n(t)| \leq \frac{\varepsilon^2}{\varepsilon} = \varepsilon$$

so

$$u^*_n(t)u_n(t) \longrightarrow 0 \quad \text{uniformly in } t \in (0, T)$$

and thus

$$\int_0^T u^*_n(t)u_n(t)\mathrm{d}t \longrightarrow 0. \tag{3.29}$$

Using (3.29) in (3.25), we obtain

$$\int_0^T v_n(t)u'_n(t)\mathrm{d}t \longrightarrow 0. \tag{3.30}$$

Because of hypothesis $H(a)(4)$, we have that

$$\int_0^T v_n(t)u'_n(t)\mathrm{d}t \leq p\int_0^T G(t, u'_n(t))\mathrm{d}t,$$

and from (3.30), we have

$$0 \leq \liminf_{n\to+\infty} \int_0^T G(t, u'_n(t))\mathrm{d}t. \tag{3.31}$$

On the other hand since $v_n(t) \in a(t, u'_n(t)) = \partial G(t, u'_n(t))$ for almost all $t \in (0, T)$, from the definition of the convex subdifferential, we have

$$v_n(t)u'_n(t) \geq G(t, u'_n(t)) \quad \text{for a.a. } t \in (0, T)$$

so from (3.30), we have

$$\limsup_{n\to+\infty} \int_0^T G(t, u'_n(t))\mathrm{d}t \leq 0. \tag{3.32}$$

From (3.31) and (3.32), it follows that

$$\int_0^T G(t, u'_n(t))\mathrm{d}t \longrightarrow 0. \tag{3.33}$$

Then returning to (3.24), passing to the limit as $n \to +\infty$ and using (3.23) and (3.33), we obtain

$$c - \varepsilon \;\le\; -\int_0^T j_+(t)\mathrm{d}t \;\le\; c + \varepsilon.$$

Let $\varepsilon \searrow 0$, to conclude that $c = -\int_0^T j_+(t)\mathrm{d}t$, a contradiction. This proved that the sequence $\{u_n\}_{n\ge1} \subseteq W^{1,p}_{\mathrm{per}}((0,T))$ is bounded. Thus by passing to a subsequence if necessary, we may assume that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{weakly in } W^{1,p}_{\mathrm{per}}((0,T)),\\ u_n &\longrightarrow u \quad \text{in } C([0,T]). \end{aligned}$$

From the choice of the sequence $\{u_n\}_{n\ge1} \subseteq W^{1,p}_{\mathrm{per}}((0,T))$, we have

$$\left| \langle v_n^*, u_n - u\rangle - \int_0^T u_n^*(t)(u_n-u)(t)\mathrm{d}t \right| \;\le\; \varepsilon_n$$

with $\varepsilon_n \searrow 0$. Note that $\int_0^T u_n^*(t)(u_n-u)(t)\mathrm{d}t \longrightarrow 0$ (see hypothesis $H(j)_1(3)$). So it follows that

$$\langle v_n^*, u_n - u\rangle \;\longrightarrow\; 0.$$

Invoking Lemma 3.3, we obtain that

$$u_n \;\longrightarrow\; u \quad \text{in } W^{1,p}_{\mathrm{per}}((0,T)).$$

The argument is similar if we assume that $\xi < 0$. Now instead of j_+, we use j_-. So finally we have that φ satisfies the nonsmooth Cerami condition at any level $c \neq -\int_0^T j_\pm(t)\mathrm{d}t$. □

Now we are ready for our first multiplicity result.

Theorem 3.5 *If hypotheses $H(a)$ and $H(j)_1$ hold, then problem* (1.1) *has at least two nontrivial solutions $u_0, y_0 \in C^1_{\mathrm{per}}([0,T])$.*

Proof By virtue of hypotheses $H(a)(5)$, $H(j)_1(3)$ and (4), the energy functional φ is bounded below. Consider the open set

$$U_+ \;=\; \left\{ u \in W^{1,p}_{\mathrm{per}}((0,T)) : \int_0^T |u(t)|^{p-2}u(t)\mathrm{d}t > 0 \right\}$$

and let $m_+ = \inf\limits_{\overline{U}_+} \varphi$. Because $G(t,0) = 0$ for all $t \in (0,T)$ and $j(t,0) = 0$ for almost all $t \in (0,T)$ (see Remark 3.2), we have

$$m_+ \;\le\; \varphi(0) \;=\; 0.$$

If $m_+ = \varphi(0) = 0$, then from hypothesis $H(j)_1(5)$ (the local sign condition), for every $\xi \in (0,\delta)$, we have

$$\varphi(\xi) \;=\; m_+.$$

Note that for every $\xi \in (0,\delta)$, we have $\xi \in \operatorname{int} W^{1,p}_{\mathrm{per}}((0,T))_+$ and so ξ is a local minimizer of φ, hence $0 \in \partial\varphi(\xi)$. Therefore we have produced a continuum of nonzero, constant solutions of problem (1.1).

Next suppose that $m_+ < 0 = \varphi(0)$. Because $-\int_0^T j_\pm(t)\mathrm{d}t \geq 0$ (see hypothesis $H(j)_1(4)$), from Lemma 3.4, it follows that φ satisfies the nonsmooth Cerami condition at level m_+. Let $\varphi_+ : W^{1,p}_{\mathrm{per}}((0,T)) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi_+(u) = \begin{cases} \varphi(u), & \text{if } u \in \overline{U}_+, \\ +\infty, & \text{otherwise.} \end{cases}$$

Evidently φ_+ is proper, lower semicontinuous and bounded below. Using Theorem 2.1, we can find a sequence $\{u_n\}_{n\geq 1} \subseteq U_+$, such that

$$\varphi_+(u_n) = \varphi(u_n) \searrow m_+$$

and

$$\varphi_+(u_n) \leq \varphi_+(y) + \frac{\|u_n - y\|}{n(1+\|u_n\|)}, \quad \forall y \in W^{1,p}_{\mathrm{per}}((0,T)).$$

Let $\lambda > 0$ and $h \in W^{1,p}_{\mathrm{per}}((0,T))$ and set $y = u_n + \lambda h$. Since $u_n \in U_+$, we can find $\widehat{\delta} > 0$, small enough so that

$$y = u_n + \lambda h \in \overline{U}_+, \quad \forall \lambda \in (0, \widehat{\delta}].$$

Therefore, we have

$$-\frac{\lambda\|h\|}{n(1+\|u_n\|)} \leq \varphi_+(u_n + \lambda h) - \varphi_+(u_n) = \varphi(u_n + \lambda h) - \varphi(u_n)$$

so

$$-\frac{\|h\|}{n(1+\|u_n\|)} \leq \frac{\varphi(u_n+\lambda h) - \varphi(u_n)}{\lambda}, \quad \forall \lambda \in (0, \widehat{\delta}]$$

and thus

$$-\frac{\|h\|}{n(1+\|u_n\|)} \leq \varphi^0(u_n; h), \quad \forall h \in W^{1,p}_{\mathrm{per}}((0,T)), \ \ n \geq 1.$$

Using Theorem 2.2, there exists $w_n^* \in W^{1,p}_{\mathrm{per}}((0,T))^*$ with $\|w_n^*\|_* = 1$, such that

$$\langle w_n^*, h\rangle \leq n(1+\|u_n\|)\varphi^0(u_n; h), \quad \forall h \in W^{1,p}_{\mathrm{per}}((0,T))$$

so

$$\frac{w_n^*}{n(1+\|u_n\|)} \in \partial\varphi(u_n), \quad \forall n \geq 1$$

and thus

$$(1+\|u_n\|)m^{\varphi}(u_n) \leq \frac{1}{n} \longrightarrow 0.$$

Thus by Lemma 3.4, we can say that

$$u_n \longrightarrow u_0 \quad \text{in } W^{1,p}_{\mathrm{per}}((0,T)).$$

We have that $u_0 \in \overline{U}_+$ and

$$m_+ = \varphi_+(u_0) = \varphi(u_0).$$

Suppose that $u_0 \in \partial U_+$. Then

$$\int_0^T |u_0(t)|^{p-2} u_0(t)\mathrm{d}t \;=\; 0.$$

Moreover, from hypothesis $H(a)(5)$ and $H(j)_1(6)$ and the variational characterization of $\lambda_1 > 0$ (see (2.2)), we have that

$$\begin{aligned} 0 > m_+ &= \int_0^T G(t, u_0'(t))\mathrm{d}t - \int_0^T j(t, u_0(t))\mathrm{d}t \\ &\geq c_0 \|u_0'\|_p^p - c_0\lambda_1 \|u_0\|_p^p \\ &\geq c_0 \|u_0'\|_p^p - c_0 \|u_0'\|_p^p \;=\; 0 \end{aligned}$$

a contradiction. So $u_0 \in U_+$. Hence $u_0 \neq 0$ is a local minimizer of φ and for this reason we have that $0 \in \partial\varphi(u_0)$. This inclusion implies that we can find $v_0^* \in A(u_0)$ and $u_0^* \in S^{p'}_{\partial j(\cdot, u_0(\cdot))}$, such that $v_0^* = u_0^*$. By definition $v_0^* = -v_0'$ with $v_0 \in S^{p'}_{a(\cdot, u_0'(\cdot))}$. Let $\langle \cdot, \cdot \rangle$ be the duality brackets for the pair $\big(W^{1,p}_{\mathrm{per}}((0,T)), W^{1,p}_{\mathrm{per}}((0,T))^*\big)$. For every $\vartheta \in C_c^1((0,T))$, we have

$$\langle v_0^*, \vartheta \rangle \;=\; \int_0^T u_0^*(t)\vartheta(t)\mathrm{d}t,$$

so

$$\int_0^T v_0(t)\vartheta'(t)\mathrm{d}t \;=\; \int_0^T u_0^*(t)\vartheta(t)\mathrm{d}t$$

and thus

$$\langle -v_0', \vartheta \rangle \;=\; \langle u_0^*, \vartheta \rangle.$$

Because the embedding $C_c^1((0,T)) \subseteq W^{1,p}_{\mathrm{per}}((0,T))$ is dense, from the last equality and since $\vartheta \in C_c^1((0,T))$ was arbitrary, we infer that

$$\begin{aligned} & v_0^*(t) \;=\; -v_0'(t) \;=\; u_0^*(t) \quad \text{for a.a. } t \in (0,T), \\ & u_0(0) = u_0(T) \end{aligned} \tag{3.34}$$

with $v_0 \in S^{p'}_{a(\cdot, u_0'(\cdot))}$, $u_0^* \in S^{p'}_{\partial j(\cdot, u_0(\cdot))}$. Evidently $v_0 \in W^{1,p'}((0,T)) \subseteq C([0,T])$ and we have

$$u_0'(t) \;=\; a^{-1}(t, v_0(t)), \quad \forall t \in (0,T).$$

By virtue of hypothesis $H(a)(2)$, the function $(t,v) \longmapsto a^{-1}(t,v)$ is single valued. We claim that this map is continuous. To this end suppose that $\{(t_n, v_n)\}_{n\geq 1} \subseteq (0,T) \times \mathbb{R}$ is a sequence, such that

$$(t_n, v_n) \;\longrightarrow\; (t, v_0) \quad \text{in } (0,T) \times \mathbb{R}$$

and

$$y_n \;=\; a^{-1}(t_n, v_n), \quad \forall n \geq 1.$$

From the definition of the convex subdifferential, hypothesis $H(a)(5)$ and since $G(t,0) = 0$ for all $t \in (0,T)$, we have

$$v_n y_n \;\geq\; G(t_n, y_n) \;\geq\; c_0 |y_n|^p$$

so

$$|y_n|^{p-1} \leq \frac{1}{c_0}|v_n| \quad \forall n \geq 1.$$

It follows that the sequence $\{y_n\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and, passing to a subsequence if necessary, we may assume that

$$y_n \longrightarrow y \quad \text{in } \mathbb{R}.$$

Again form the definition of the convex subdifferential, we have that

$$v_n(z - y_n) \leq G(t_n, z) - G(t_n, y_n), \quad \forall z \in \mathbb{R}$$

so

$$v_0(z - y) \leq G(t, z) - G(t, y), \quad \forall z \in \mathbb{R},$$

thus

$$v_0 \in \partial G(t, y) = a(t, y)$$

and so $y = a^{-1}(t, v_0)$. This proves that indeed the map $(t, v) \longmapsto a^{-1}(t, v)$ is continuous on $(0, T) \times \mathbb{R}$. Hence the map $t \longmapsto a^{-1}(t, v_0(t)) = u_0'(t)$ is continuous and so $u_0 \in C^1([0, T])$. Using integration by parts, for every $\eta \in W^{1,p}_{\text{per}}((0, T))$, we have

$$\langle v_0^*, \eta\rangle = \int_0^T v_0(t)\eta'(t)\mathrm{d}t = \int_0^T u_0^*(t)\eta(t)\mathrm{d}t$$

with $v_0 \in S^{p'}_{a(\cdot, u_0'(\cdot))}$, so

$$v_0(T)\eta(T) - v_0(0)\eta(0) - \int_0^T v_0'(t)\eta(t)\mathrm{d}t = \int_0^T u_0^*(t)\eta(t)\mathrm{d}t$$

and thus

$$v_0(0)\eta(0) = v_0(T)\eta(T).$$

Since $\eta \in W^{1,p}_{\text{per}}((0, T))$ was arbitrary, it follows that $v_0(0) = v_0(T)$. Then because of hypothesis $H(a)(2)$, we have that

$$u_0'(0) = a^{-1}(0, v_0(0)) = a^{-1}(T, v_0(T)) = u_0'(T)$$

so $u_0 \in C^1_{\text{per}}([0, T])$ is a nontrivial solution for problem (1.1).

Considering the open set $U_- \subseteq W^{1,p}_{\text{per}}((0, T))$, defined by

$$U_- = \left\{u \in W^{1,p}_{\text{per}}((0, T)) : \int_0^T |u(t)|^{p-2}u(t)\mathrm{d}t < 0\right\}$$

and arguing as before (with U_+ replaced by U_-), we obtain another solution $y_0 \in U_-$ of (1.1), with $y_0 \neq 0$, $y_0 \neq u_0$.

This way we have produced two distinct nonzero solutions for problem (1.1). □

Remark 3.6 An example of a nonsmooth function satisfying hypotheses $H(j)_1$ is the following (for simplicity we drop the t-dependence):

$$j(\zeta) = \begin{cases} ec_0\lambda_1 e^{\zeta}, & \text{if } \zeta < -1, \\ c_0\lambda_1|\zeta|^p, & \text{if } |\zeta| \le 1, \\ \dfrac{c_0\lambda_1}{\sqrt{\zeta}}, & \text{if } \zeta > 1. \end{cases}$$

Another function satisfying hypotheses $H(j)_1$ is the following:

$$j(\zeta) = \begin{cases} \arctan(\zeta + 1), & \text{if } \zeta < -1, \\ 0, & \text{if } \zeta \in [-1, 0], \\ \dfrac{c_0\lambda_1\zeta^p}{a^{\zeta}}, & \text{if } \zeta > 0 \end{cases}$$

with $a > 1$.

4 Existence of three solutions

We can guarantee the existence of three solutions, by modifying our hypotheses on the nonsmooth potential. More precisely our new hypotheses on j are the following.

$H(j)_2$ j: $(0, T) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(1) for every $\zeta \in \mathbb{R}$, the function $t \longrightarrow j(t, \zeta)$ is measurable;
(2) for almost all $t \in (0, T)$, the function $\zeta \longmapsto j(t, \zeta)$ is locally Lipschitz with $L^{p'}((0, T))_+$-Lipschitz constant;
(3) for every $M > 0$, there exists $\widehat{a}_M \in L^1((0, T))_+$, such that for almost all $t \in (0, T)$, all $|\zeta| \le M$ and all $u^* \in \partial j(t, \zeta)$, we have $|u^*| \le \widehat{a}_M(t)$;
(4) there exist $j_\pm \in L^1((0, T))$, such that

$$\lim_{\zeta \to \pm\infty} j(t, \zeta) = j_\pm(t),$$

uniformly for almost all $t \in (0, T)$;
(5) for almost all $t \in (0, T)$ and all $\zeta \in \mathbb{R}$, we have

$$j(t, \zeta) \le c_0\lambda_1|\zeta|^p,$$

with $c_0 > 0$ as in hypothesis $H(a)(5)$ and $\lambda_1 > 0$ being the first nonzero eigenvalue of the negative p-Laplacian with periodic boundary condition;
(6) there exist $\xi_- < 0 < \xi_+$ such that

$$\int_0^T j(t, \xi_\pm)\mathrm{d}t > 0 > \int_0^T j_\pm(t)\mathrm{d}t.$$

Remark 4.1 Note that the strong resonance hypothesis $H(j)_2(4)$ is still in effect. We no longer impose the local sign condition (see hypothesis $H(j)_1(5)$). Instead we employ hypothesis $H(j)_2(6)$.

A careful reading of the proof of Lemma 3.4, reveals that the result remains valid in the present situation, namely the energy functional φ satisfies the nonsmooth Cerami condition at any level $c \ne -\int_0^T j_\pm(t)\mathrm{d}t$. Then we can prove the following three solutions theorem.

Theorem 4.2 *If hypotheses $H(a)$ and $H(j)_2$ hold, then problem* (1.1) *has at least three solutions $u_0, y_0, z_0 \in C^1_{\text{per}}([0,T])$.*

Proof Using the sets $U_\pm \subseteq W^{1,p}_{\text{per}}((0,T))$, as in the proof of Theorem 3.5, we can produce two nontrivial solutions $u_0, y_0 \in W^{1,p}_{\text{per}}((0,T))$, $u_0 \in U_+$, $y_0 \in U_-$. Note that in the present setting, it cannot happen that $m_\pm = 0$, since

$$m_+ \le -\int_0^T j(t,\xi_+)\mathrm{d}t < 0 \qquad \text{and} \quad m_- \le -\int_0^T j(t,\xi_-)\mathrm{d}t < 0$$

(see hypothesis $H(j)_2(6)$).

Next let

$$\begin{aligned} E_1 &= \{\xi_+, \xi_-\} \\ E &= [\xi_-, \xi_+] == \left\{u \in W^{1,p}_{\text{per}}((0,T)) : \ \xi_- \le u(t) \le \xi_+ \ \text{ for all } t \in (0,T)\right\}, \\ D &= \left\{u \in W^{1,p}_{\text{per}}((0,T)) : \int_0^T |u(t)|^{p-2}u(t)\mathrm{d}t = 0\right\}. \end{aligned}$$

We claim that E_1 and D link in $W^{1,p}_{\text{per}}((0,T))$. Indeed, first note that $E_1 \cap D = \emptyset$. Next let $\vartheta \in C\big(E; W^{1,p}_{\text{per}}((0,T))\big)$, with $\vartheta|_{E_1} = id|_{E_1}$, i.e. $\vartheta(\xi_-) = \xi_-$ and $\vartheta(\xi_+) = \xi_+$. Let $\psi \colon W^{1,p}_{\text{per}}((0,T)) \longrightarrow \mathbb{R}$ be defined by

$$\psi(u) = \int_0^T |u(t)|^{p-2}u(t)\mathrm{d}t.$$

Then $\psi \in C\big(W^{1,p}_{\text{per}}((0,T))\big)$ and so $\psi \circ \vartheta \in C(E)$. We have

$$(\psi \circ \vartheta)(\xi_-) = \psi(\xi_-) < 0 < \psi(\xi_+) = (\psi \circ \vartheta)(\xi_+).$$

Evidently E is connected. Hence so is $(\psi \circ \vartheta)(E)$ and so we can find $u \in E$, such that $(\psi \circ \vartheta)(u) = 0$. We have $\psi(\vartheta(u)) = 0$, which means that $\vartheta(u) \in D$. Therefore $\vartheta(E) \cap D \ne \emptyset$, which proves that the two sets E_1 and D link in $W^{1,p}_{\text{per}}((0,T))$. Applying Theorem 2.3, we obtain $z_0 \in W^{1,p}_{\text{per}}((0,T))$, such that

$$\varphi(z_0) \ge \inf_D \varphi = 0 > m_\pm \quad \text{and} \quad 0 \in \partial\varphi(z_0).$$

Since $m_+ = \varphi(u_0)$, $m_- = \varphi(y_0)$, we see that $z_0 \ne u_0$ and $z_0 \ne y_0$ and from the inclusion $0 \in \partial\varphi(z_0)$, it follows that $z_0 \in C^1_{\text{per}}([0,T])$ is a third solution of problem (1.1). □

Remark 4.3 A nonsmooth potential satisfying hypothesis $H(j)_2$ is given by the following function (again for simplicity we drop the t-dependence):

$$j(\zeta) = \begin{cases} \dfrac{2c_0\lambda_1}{\sqrt{|\zeta|}} - c_0\lambda_1, & \text{if} \quad \zeta < -1, \\ c_0\lambda_1|\zeta|^p, & \text{if} \quad \zeta \in [-1,0], \\ \zeta \ln \zeta, & \text{if} \quad \zeta \in (0,1], \\ \dfrac{c\ln\zeta}{\zeta} - \arctan(\zeta - 1), & \text{if} \quad \zeta > 1 \end{cases}$$

with $c > 1$.

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